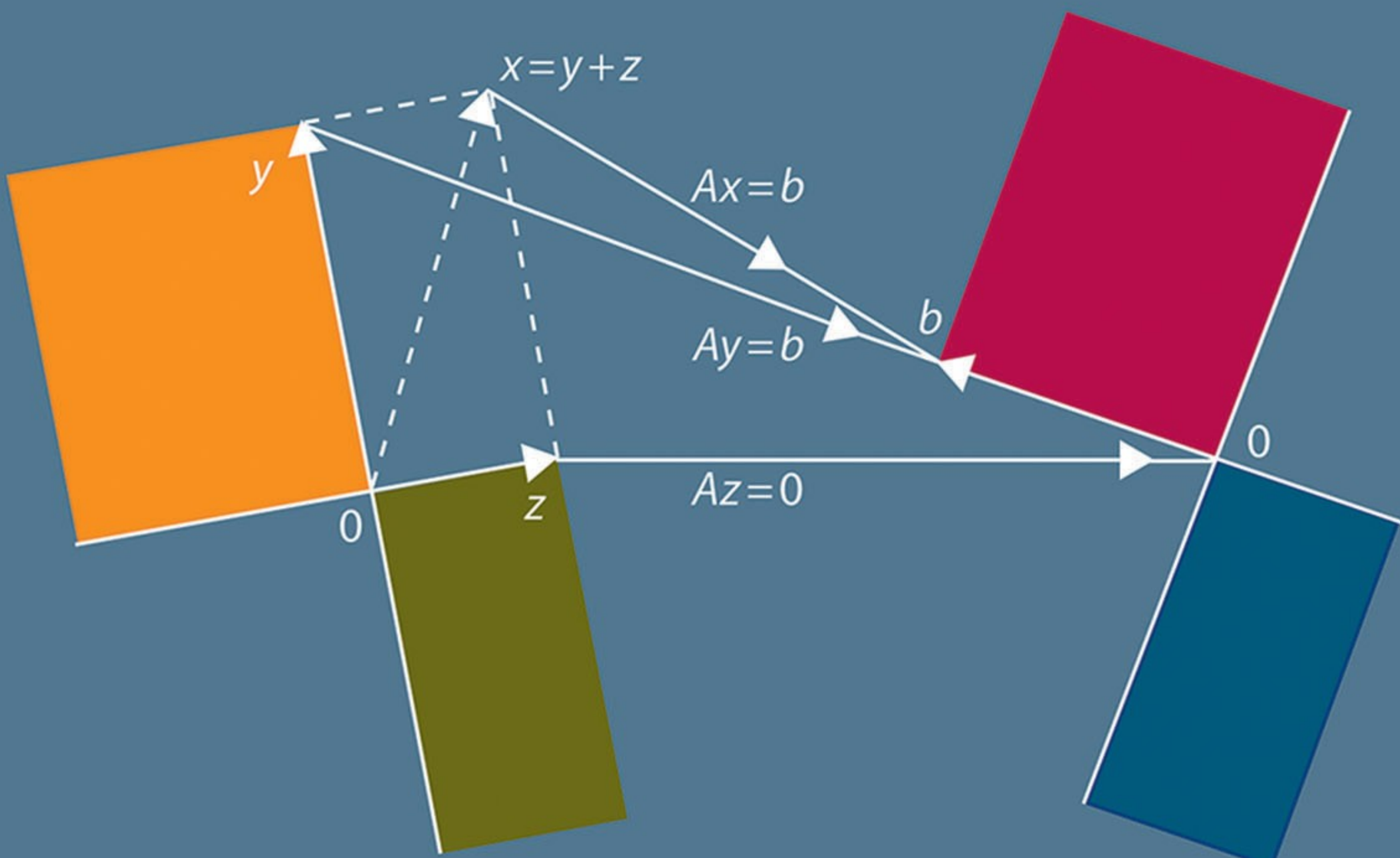
Introduction to

LINEAR ALGEBRA

SIXTH EDITION



GILBERT STRANG

INTRODUCTION TO LINEAR ALGEBRA

Sixth Edition

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WELLESLEY - CAMBRIDGE PRESS
Box 812060 Wellesley MA 02482

Introduction to Linear Algebra, 6th Edition

ISBN 978-1-7331466-7-8

LaTeX typesetting by Ashley C. Fernandes (info@problemsolvingpathway.com)
Printed in the United States of America

9 8 7 6 5 4 3 2 1

QA184.S78_2023

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Wellesley - Cambridge Press
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For orders: **math.mit.edu/weborder.php**
Outside US/Canada: **www.cambridge.org**
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The website for this book (with Solution Manual) is **math.mit.edu/linearalgebra**
2020 book: **Linear Algebra for Everyone (math.mit.edu/everyone)**
2019 book: **Linear Algebra and Learning from Data (math.mit.edu/learningfromdata)**
2014 book: **Differential Equations and Linear Algebra (math.mit.edu/dela)**

Linear Algebra (Math 18.06) is included in MIT's OpenCourseWare **ocw.mit.edu/courses**
Those videos (including 18.06SC and 18.065) are also on **www.youtube.com/mitocw**
18.06 Linear Algebra   18.06SC with problem solutions   18.065 Learning from Data

The cover design was created by Gail Corbett and Lois Sellers: **lsellersdesign.com**

Kenji Hiranabe contributed **The Five Factorizations** (back cover) and **Matrix World**

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Preface

One goal of this Preface can be achieved right away. You need to know about the video lectures for MIT's Linear Algebra course **Math 18.06**. Those videos go with this book, and they are part of MIT's OpenCourseWare. The direct links to linear algebra are

https://ocw.mit.edu/courses/18-06-linear-algebra-spring-2010/

https://ocw.mit.edu/courses/18-06sc-linear-algebra-fall-2011/

On YouTube those lectures are at **https://ocw.mit.edu/1806videos** and **/1806scvideos**

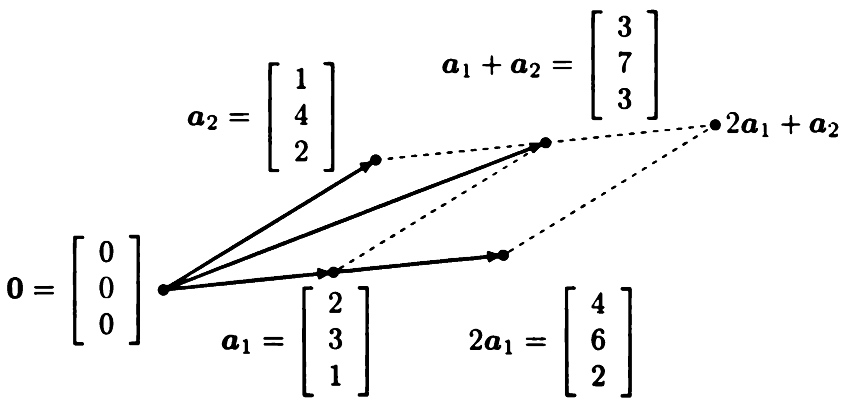
The first link brings the original lectures from the dawn of OpenCourseWare. Problem solutions by graduate students (really good) and also a short introduction to linear algebra were added to the new 2011 lectures. And the course today has a new start—the crucial ideas of linear independence and the column space of a matrix have moved near the front.

I would like to tell you about those ideas in this Preface.

Start with two column vectors $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$. They can have three components each, so they correspond to points in 3-dimensional space. The picture needs a center point which locates the zero vector:

$$\boldsymbol{a}_1 = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} \qquad \boldsymbol{a}_2 = \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} \qquad \textbf{zero vector} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

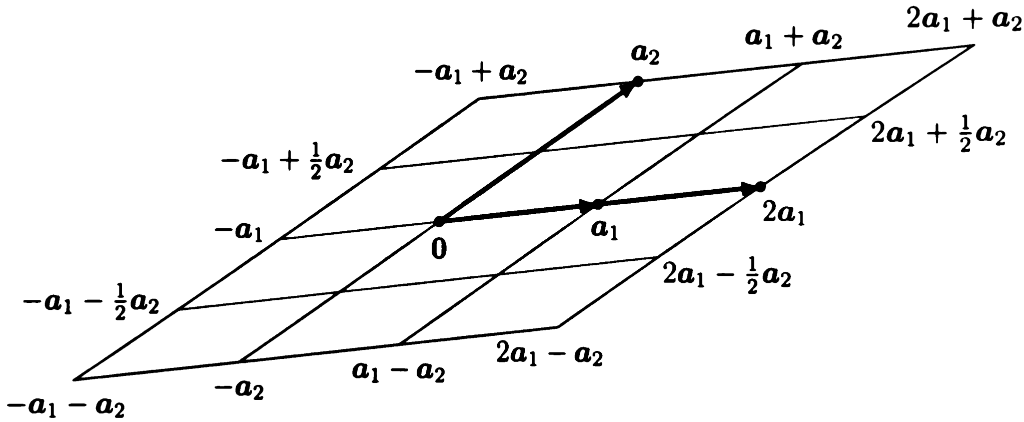
The vectors are drawn on this 2-dimensional page. But we all have practice in visualizing three-dimensional pictures. Here are $\boldsymbol{a}_1, \boldsymbol{a}_2, 2\boldsymbol{a}_1$, and the vector sum $\boldsymbol{a}_1 + \boldsymbol{a}_2$.



That picture illustrated two basic operations—adding vectors $\boldsymbol{a}_1 + \boldsymbol{a}_2$ and multiplying a vector by 2. Combining those operations produced a **"linear combination"** $2\boldsymbol{a}_1 + \boldsymbol{a}_2$:

$$\boxed{\textbf{Linear combination} = c\boldsymbol{a}_1 + d\boldsymbol{a}_2 \ \textbf{ for any numbers } c \textbf{ and } d}$$

Those numbers c and d can be negative. In that case $c\boldsymbol{a}_1$ and $d\boldsymbol{a}_2$ will reverse their directions: they go right to left. Also very important, c and d can involve fractions. Here is a picture with a lot more linear combinations. **Eventually we want all vectors** $c\boldsymbol{a}_1 + d\boldsymbol{a}_2$.



Here is the key! **The combinations $c\boldsymbol{a}_1 + d\boldsymbol{a}_2$ fill a whole plane.** It is an infinite plane in 3-dimensional space. By using more and more fractions and decimals c and d, we fill in a complete plane. *Every point on the plane is a combination of* $\boldsymbol{a}_1$ *and* $\boldsymbol{a}_2$.

Now comes a fundamental idea in linear algebra: **a matrix**. The matrix A holds **n column vectors** $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n$. At this point our matrix has two columns $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$, and those are vectors in 3-dimensional space. So the matrix has **three rows and two columns**.

3 by 2 matrix $m = 3$ **rows** $n = 2$ **columns**	$A = \begin{bmatrix} \\ \boldsymbol{a}_1 & \boldsymbol{a}_2 \\ \\ \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 3 & 4 \\ 1 & 2 \end{bmatrix}$

The combinations of those two columns produced a plane in three-dimensional space. There is a natural name for that plane. **It is the column space of the matrix.** For any A, **the column space of A contains all combinations of the columns.**

Here are the four ideas introduced so far. You will see them all in Chapter 1.

1. **Column vectors** $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ in three dimensions

2. **Linear combinations** $c\boldsymbol{a}_1 + d\boldsymbol{a}_2$ of those vectors

3. **The matrix A** contains the columns $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$

4. **Column space of the matrix** = *all linear combinations of the columns = plane*

Now we include 2 more columns in A

The 4 columns are in 3-dimensional space

$$A = \begin{bmatrix} \mathbf{2} & \mathbf{1} & \mathbf{3} & \mathbf{0} \\ \mathbf{3} & \mathbf{4} & \mathbf{7} & \mathbf{0} \\ \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{-1} \end{bmatrix}$$

Linear algebra aims for an understanding of every column space. Let me try this one.

Columns 1 and 2 produce the same plane as before (same $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$)

Column 3 contributes nothing new **because $\boldsymbol{a}_3$ is on that plane : $\boldsymbol{a}_3 = \boldsymbol{a}_1 + \boldsymbol{a}_2$**

Column 4 is **not on the plane** : Adding in $c_4\boldsymbol{a}_4$ raises or lowers the plane

The column space of this matrix A is the **whole 3-dimensional space** : all points !

You see how we go a column at a time, left to right. Each column can be **independent** of the previous columns or it can be a **combination** of those columns. To produce every point in 3-dimensional space, you need three independent columns.

Matrix Multiplication $A = CR$

Using the words "linear combination" and "independent columns" gives a good picture of that 3 by 4 matrix A. Column **3** is a linear combination: **column 1 + column 2. Columns $\mathbf{1, 2, 4}$ are independent.** The only way to produce the zero vector as a combination of the independent columns 1, 2, 4 is to multiply all those columns by **zero.**

We are so close to a key idea of Chapter 1 that I have to go on. Matrix multiplication is the perfect way to write down what we know. From the 4 columns of A we pick out the independent columns $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_4$ in the column matrix C. **Every column of R tells us the combination of $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_4$ in C that produces a column of A. A equals C times R** :

$$A = \begin{bmatrix} 2 & 1 & \mathbf{3} & 0 \\ 3 & 4 & \mathbf{7} & 0 \\ 1 & 2 & \mathbf{3} & -1 \end{bmatrix} = \begin{bmatrix} \mathbf{2} & \mathbf{1} & 0 \\ \mathbf{3} & \mathbf{4} & 0 \\ \mathbf{1} & \mathbf{2} & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & \mathbf{1} & 0 \\ 0 & 1 & \mathbf{1} & 0 \\ 0 & 0 & \mathbf{0} & 1 \end{bmatrix} = CR$$

Column 3 of A is dependent on columns 1 and 2 of A, and column 3 of R shows how. Add the independent columns 1 and 2 of C to get column $\boldsymbol{a}_3 = \boldsymbol{a}_1 + \boldsymbol{a}_2 = (\mathbf{3}, \mathbf{7}, \mathbf{3})$ of A.

Matrix multiplication : Each column j of CR is C times column j of R

Section 1.3 of the book will *multiply a matrix times a vector* (two ways). Then Section 1.4 will *multiply a matrix times a matrix.* This is the key operation of linear algebra. It is important that there is more than one good way to do this multiplication.

I am going to stop here. The normal purpose of the Preface is to tell you about the big picture. The next pages will give you two ways to organize this subject—especially the first seven chapters that more than fill up most linear algebra courses. Then come optional chapters, leading to the most active topic in applications today : **deep learning.**

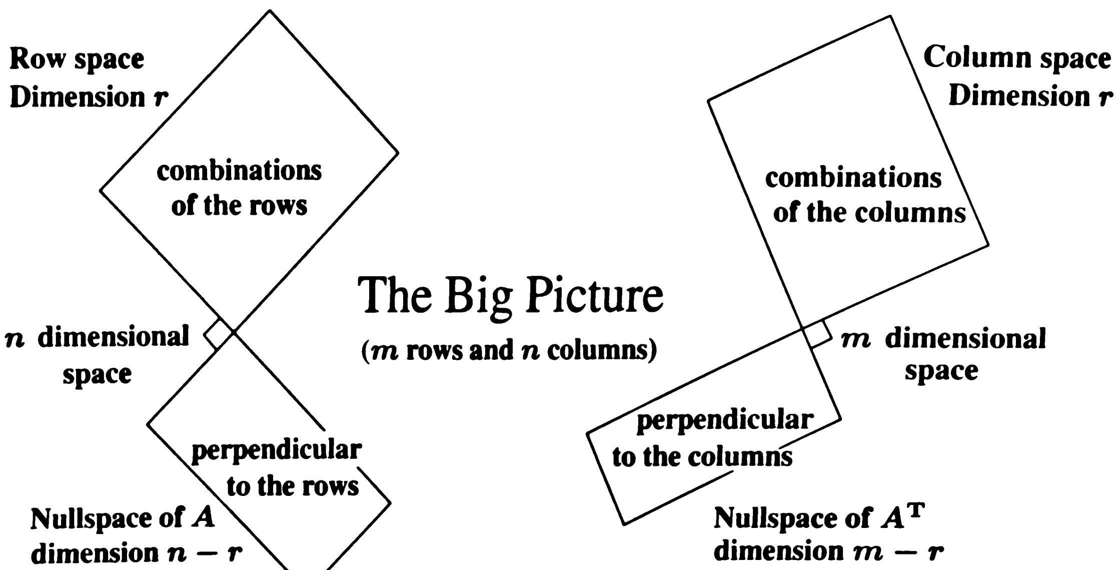
The Four Fundamental Subspaces

You have just seen how the course begins—with the columns of a matrix A. There were two key steps. One step was to take all combinations $c\boldsymbol{a}_1 + d\boldsymbol{a}_2 + e\boldsymbol{a}_3 + f\boldsymbol{a}_4$ of the columns. This led to the **column space of A**. The other step was to **factor the matrix into C times R**. That matrix C holds a full set of **independent columns**.

I fully recognize that this is only the Preface to the book. You have had zero practice with the column space of a matrix (and even less practice with C and R). But the good thing is: Those are the right directions to start. Eventually, *every matrix will lead to four fundamental spaces*. Together with the column space of A comes the **row space—all combinations of the rows**. When we take all combinations of the n columns and all combinations of the m rows—those combinations fill up "spaces" of vectors.

The other two subspaces complete the picture. Suppose the row space is a plane in three dimensions. Then there is one special direction in the 3D picture—**that direction is perpendicular to the row space**. That perpendicular line is the **nullspace** of the matrix. We will see that the vectors in the nullspace (perpendicular to all the rows) solve $A\boldsymbol{x} = \boldsymbol{0}$: the most basic of linear equations.

And if vectors perpendicular to all the rows are important, so are the vectors perpendicular to all the columns. Here is the picture of the **Four Fundamental Subspaces**.



The Four Fundamental Subspaces: An m by n matrix with r independent columns.

This picture of four subspaces comes in Chapter 3. The idea of perpendicular spaces is developed in Chapter 4. And special "basis vectors" for all four subspaces are discovered in Chapter 7. That step is the final piece in the **Fundamental Theorem of Linear Algebra**. The theorem includes an amazing fact about any matrix, square or rectangular:

The number of independent columns equals the number of independent rows.

Five Factorizations of a Matrix

Here are the organizing principles of linear algebra. When our matrix has a special property, these factorizations will show it. Chapter after chapter, they express the key idea in a direct and useful way.

The usefulness increases as you go down the list. **Orthogonal matrices** are the winners in the end, because their columns are perpendicular unit vectors. That is perfection.

$$\text{2 by 2 Orthogonal Matrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \text{Rotation by Angle } \theta$$

Here are the five factorizations from Chapters 1, 2, 4, 6, 7 :

1	$\boldsymbol{A = CR}$	$=$ $\boldsymbol{R}$ combines independent columns in C to give all columns of $\boldsymbol{A}$
2	$\boldsymbol{A = LU}$	$=$ Lower triangular $\boldsymbol{L}$ times Upper triangular $\boldsymbol{U}$
4	$\boldsymbol{A = QR}$	$=$ Orthogonal matrix $\boldsymbol{Q}$ times Upper triangular $\boldsymbol{R}$
6	$\boldsymbol{S = Q\Lambda Q^{\mathrm{T}}}$	$=$ (Orthogonal $\boldsymbol{Q}$) (Eigenvalues in Λ) (Orthogonal $\boldsymbol{Q}^{\mathrm{T}}$)
7	$\boldsymbol{A = U\Sigma V^{\mathrm{T}}}$	$=$ (Orthogonal U) (Singular values in Σ) (Orthogonal $\boldsymbol{V}^{\mathrm{T}}$)

May I call your attention to the last one ? It is the **Singular Value Decomposition (SVD)**. It applies to every matrix A. Those factors U and V have perpendicular columns—all of length one. Multiplying any vector by U or V leaves a vector of the same length—so computations don't blow up or down. And Σ is a positive diagonal matrix of "singular values". If you learn about eigenvalues and eigenvectors in Chapter 6, *please* continue a few pages to singular values in Section 7.1.

Deep Learning

For a true picture of linear algebra, applications have to be included. Completeness is totally impossible. At this moment, the dominating direction of applied mathematics has one special requirement: *It cannot be entirely linear* !

One name for that direction is "deep learning". It is an extremely successful approach to a fundamental scientific problem: **Learning from data**. In many cases the data comes in a matrix. Our goal is to look inside the matrix for the connections between variables. Instead of solving matrix equations or differential equations that express known input-output rules, *we have to find those rules*. The success of deep learning is to build a function $\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{v})$ with inputs $\boldsymbol{x}$ and $\boldsymbol{v}$ of two kinds :

The vectors $\boldsymbol{v}$ describes the features of the training data.

The matrices $\boldsymbol{x}$ assign weights to those features.

The function $\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{v})$ is close to the correct output for that training data $\boldsymbol{v}$.

When $\boldsymbol{v}$ changes to unseen test data, $\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{v})$ stays close to correct.

This success comes partly from the form of the learning function $\boldsymbol{F}$, which allows it to include vast amounts of data. In the end, a linear function $\boldsymbol{F}$ would be totally inadequate. The favorite choice for $\boldsymbol{F}$ is **piecewise linear**. This combines simplicity with generality.

Applications in the Book and on the Website

I hope this book will be useful to you long after the linear algebra course is complete. It is all the applications of linear algebra that make this possible. Matrices carry data, and other matrices *operate on that data*. The goal is to "see into a matrix" by understanding its eigenvalues and eigenvectors and singular values and singular vectors. And each application has special matrices—here are four examples:

Markov matrices M	Each column is a set of probabilities adding to 1.
Incidence matrices A	Graphs and networks start with a set of nodes. The matrix A tells the *connections* (edges) between those nodes.
Transform matrices F	The Fourier matrix uncovers the *frequencies* in the data.
Covariance matrices C	The **variance** is key information about a random variable. *The covariance explains dependence between variables.*

We included those applications and more in this Sixth Edition. For the crucial computation of matrix weights in deep learning, Chapter 9 presents the ideas of **optimization**. This is where linear algebra meets calculus: **derivative = zero** becomes a matrix equation at the minimum point because $F(x)$ has many variables.

Several topics from the Fifth Edition gave up their places but not their importance. Those sections simply moved onto the Web. The website for this new Sixth Edition is

math.mit.edu/linearalgebra

That website includes sample sections from this new edition and solutions to all Problem Sets. These sections (and more) are saved online from the Fifth Edition:

Fourier Series | **Norms and Condition Numbers**

Iterative Methods and Preconditioners | **Linear Algebra for Cryptography**

Here is a small touch of linear algebra—three questions before this course gets serious:

1. Suppose you draw three straight line segments of lengths r and s and t on this page. What are the conditions on those three lengths to allow you to make the segments into a triangle? In this question you can choose the directions of the three lines.

2. Now suppose the directions of three straight lines u, v, w are fixed and different. But you could stretch those lines to au, bv, cw with any numbers a, b, c. Can you always make a closed triangle out of the three vectors au, bv, cw?

3. Linear algebra doesn't stay in a plane! Suppose you have **four lines** u, v, w, z in different directions in 3-dimensional space. Can you always choose the numbers a, b, c, d (zeros not allowed) so that $au + bv + cw + dz = 0$?

For typesetting this book, maintaining its website, offering quality textbooks to Indian fans, I am grateful to Ashley C. Fernandes of Wellesley Publishers (www.wellesleypublishers.com)

gilstrang@gmail.com Gilbert Strang

1 Vectors and Matrices

Linear algebra is about vectors $\boldsymbol{v}$ and matrices A. Those are the basic objects that we can add and subtract and multiply (when their shapes match correctly). The first vector $\boldsymbol{v}$ has two components $\boldsymbol{v_1 = 2}$ and $\boldsymbol{v_2 = 4}$. The vector $\boldsymbol{w}$ is also 2-dimensional.

$$\boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} \qquad \boldsymbol{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \qquad \boldsymbol{v} + \boldsymbol{w} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$$

The **linear combinations of $\boldsymbol{v}$ and $\boldsymbol{w}$ are the vectors** $c\boldsymbol{v} + d\boldsymbol{w}$ for all numbers c and d:

$$\textbf{The linear combinations} \quad c\begin{bmatrix} 2 \\ 4 \end{bmatrix} + d\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2c + 1d \\ 4c + 3d \end{bmatrix} \quad \textbf{fill the } \boldsymbol{xy} \textbf{ plane}$$

The **length** of that vector $\boldsymbol{w}$ is $||\boldsymbol{w}|| = \sqrt{10}$, the square root of $w_1^2 + w_2^2 = 1 + 9$. The **dot product** of $\boldsymbol{v}$ and $\boldsymbol{w}$ is $\boldsymbol{v} \cdot \boldsymbol{w} = v_1 w_1 + v_2 w_2 = (2)(1) + (4)(3) = 14$. In Section 1.2, $\boldsymbol{v} \cdot \boldsymbol{w}$ will reveal the angle between those vectors.

The big step in Section 1.3 is to introduce a **matrix**. This matrix $\boldsymbol{A}$ contains our two column vectors. The vectors have two components, so the matrix is 2 by 2:

$$A = \begin{bmatrix} \boldsymbol{v} & \boldsymbol{w} \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$$

When a matrix multiplies a vector, we get a combination $c\boldsymbol{v} + d\boldsymbol{w}$ of its columns:

$$A \text{ times} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 2c + 1d \\ 4c + 3d \end{bmatrix} = c\boldsymbol{v} + d\boldsymbol{w}.$$

And when we look at **all combinations $A\boldsymbol{x}$** (with every c and d), those vectors produce the **column space of the matrix A**. Here that column space is a plane.

With three vectors, the new matrix $\boldsymbol{B}$ has columns $\boldsymbol{v}, \boldsymbol{w}, \boldsymbol{z}$. In this example $\boldsymbol{z}$ is a combination of $\boldsymbol{v}$ and $\boldsymbol{w}$. So the column space of $\boldsymbol{B}$ is still the xy plane. The vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ are **independent** but $\boldsymbol{v}, \boldsymbol{w}, \boldsymbol{z}$ are **dependent**. A combination produces zero:

$$B = \begin{bmatrix} \boldsymbol{v} & \boldsymbol{w} & \boldsymbol{z} \end{bmatrix} = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 3 & 7 \end{bmatrix} \text{ has } B\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \boldsymbol{v} + \boldsymbol{w} - \boldsymbol{z} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \end{bmatrix} - \begin{bmatrix} 3 \\ 7 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The final goal is to understand **matrix multiplication $AB = A$ times each column of B.**

1.1 Vectors and Linear Combinations

1 $2\boldsymbol{v} - 3\boldsymbol{w}$ **is a linear combination** $c\boldsymbol{v} + d\boldsymbol{w}$ of the vectors $\boldsymbol{v}$ and $\boldsymbol{w}$.

2 For $\boldsymbol{v} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ and $\boldsymbol{w} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ that combination is $2\begin{bmatrix} 4 \\ 1 \end{bmatrix} - 3\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$.

3 All combinations $c\begin{bmatrix} 4 \\ 1 \end{bmatrix} + d\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ fill the xy plane. They produce every $\begin{bmatrix} x \\ y \end{bmatrix}$.

4 The vectors $c\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + d\begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}$ fill a **plane** in xyz space. $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ **is not on that plane.**

Calculus begins with numbers x and functions $f(x)$. Linear algebra begins with vectors $\boldsymbol{v}, \boldsymbol{w}$ and their linear combinations $c\boldsymbol{v} + d\boldsymbol{w}$. Immediately this takes you into two or more (possibly many more) dimensions. But linear combinations of vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ are built from just two basic operations:

Multiply a vector $\boldsymbol{v}$ by a number $\quad 3\boldsymbol{v} = 3\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$

Add vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ of the same dimension: $\boldsymbol{v} + \boldsymbol{w} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$

Those operations come together in a **linear combination** $c\boldsymbol{v} + d\boldsymbol{w}$ of $\boldsymbol{v}$ and $\boldsymbol{w}$:

Linear combination
$c = 5$ **and** $d = -2$ $\quad 5\boldsymbol{v} - 2\boldsymbol{w} = 5\begin{bmatrix} 2 \\ 1 \end{bmatrix} - 2\begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \end{bmatrix} - \begin{bmatrix} 8 \\ 6 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

That idea of a linear combination opens up two key questions:

1 **Describe all the combinations** $c\boldsymbol{v} + d\boldsymbol{w}$. Do they fill a plane or a line?

2 **Find the numbers c and d** that produce a specific combination $c\boldsymbol{v} + d\boldsymbol{w} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

We can answer those questions here in Section 1.1. But linear algebra is not limited to 2 vectors in 2-dimensional space. As long as we stay linear, the problems can get bigger (more dimensions) and harder (more vectors). The vectors will have m components instead of 2 components. We will have n vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ instead of 2 vectors. **Those n vectors in m-dimensional space will go into the columns of an m by n matrix A:**

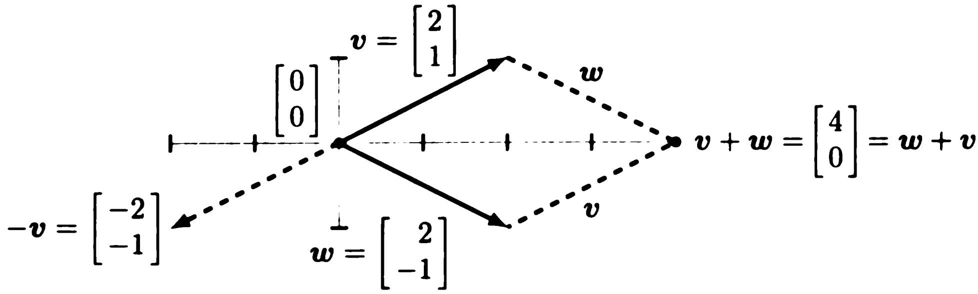
m rows
n columns $\quad A = \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \cdots & \boldsymbol{v}_n \end{bmatrix}$
m by n matrix

Let me repeat the two key questions using A, and then retreat back to $m = 2$ and $n = 2$:

1	**Describe all the combinations** $A\boldsymbol{x} = x_1\boldsymbol{v}_1 + x_2\boldsymbol{v}_2 + \cdots + x_n\boldsymbol{v}_n$ **of the columns**
2	**Find the numbers x_1 to x_n** that produce a desired output vector $A\boldsymbol{x} = \boldsymbol{b}$

Linear Combinations $cv + dw$

Start from the beginning. A vector v in 2-dimensional space has *two components*. To draw v and $-v$, use an arrow that begins at the zero vector:



The vectors cv (for all numbers c) fill an infinitely long line in the xy plane. If w is not on that line, then the vectors dw fill a second line. We aim to see that **the linear combinations $cv + dw$ fill the plane.** Combining points on the v line and the w line gives all points.

Here are four different linear combinations—*we can choose any numbers c and d*:

$$\begin{aligned} 1v + 1w &= \text{sum of vectors} \\ 1v - 1w &= \text{difference of vectors} \\ 0v + 0w &= \textit{zero vector} \\ cv + 0w &= \text{vector } cv \text{ in the direction of } v \end{aligned}$$

Solving Two Equations

Very often linear algebra offers the choice of a picture or a computation. The picture can start with $v + w$ and $w + v$ (those are equal). Another important vector is $w - v$, going backwards on v. The Preface has a picture with many more combinations—starting to fill a plane. But if we aim for a particular vector like $cv + dw = \begin{bmatrix} 8 \\ 2 \end{bmatrix}$, it will be better to compute the exact numbers c and d. Here are two ways to write down this problem.

$$\boxed{\text{Solve} \quad c\begin{bmatrix} 2 \\ 1 \end{bmatrix} + d\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \end{bmatrix}. \qquad \text{This means} \quad \begin{aligned} \mathbf{2c + 2d} &\mathbf{= 8} \\ \mathbf{c - \ \ d} &\mathbf{= 2} \end{aligned}.}$$

The rules for solution are simple but strict. We can multiply equations by numbers (*not zero!*) and we can subtract one equation from another equation. Experience teaches that the key is to produce zeros on the left side of the equations. One zero will be enough!

$\frac{1}{2}$(equation 1) is $\boldsymbol{c + d = 4}$

Subtract this from equation 2

Then c is eliminated $\longrightarrow$

$$\begin{aligned} \mathbf{2c + 2d} &\mathbf{= \ \ 8} \\ \mathbf{0c - 2d} &\mathbf{= -2} \end{aligned}$$

The second equation gives $\boldsymbol{d = 1}$. Going upwards, the first equation becomes $\mathbf{2c + 2 = 8}$. Its solution is $\boldsymbol{c = 3}$. This combination is correct:

$$\mathbf{3}\begin{bmatrix} 2 \\ 1 \end{bmatrix} + \mathbf{1}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \end{bmatrix}. \quad \text{In matrix form} \begin{bmatrix} \mathbf{2} & \mathbf{2} \\ \mathbf{1} & \mathbf{-1} \end{bmatrix}\begin{bmatrix} \mathbf{3} \\ \mathbf{1} \end{bmatrix} = \begin{bmatrix} \mathbf{8} \\ \mathbf{2} \end{bmatrix}$$

Column Way, Row Way, Matrix Way

Column way **Linear combination**	$c\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + d\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$
Row way **Two equations for c and d**	$\begin{aligned} v_1c + w_1d &= b_1 \\ v_2c + w_2d &= b_2 \end{aligned}$
Matrix way **2 by 2 matrix**	$\begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \end{bmatrix}\begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

If the points $\boldsymbol{v}$ and $\boldsymbol{w}$ and the zero vector $\mathbf{0}$ are not on the same line, there is exactly one solution c, d. Then the linear combinations of $\boldsymbol{v}$ and $\boldsymbol{w}$ exactly fill the xy plane. The vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ are **"linearly independent"**. The 2 by 2 matrix $A = \begin{bmatrix} \boldsymbol{v} & \boldsymbol{w} \end{bmatrix}$ is "*invertible*".

Can Elimination Fail?

Elimination fails to produce a solution only when the equations don't have a solution in the first place. This can happen when the vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ *lie on the same line through the center point* $(0, 0)$. Those vectors are not independent.

The reason is clear: All combinations of $\boldsymbol{v}$ and $\boldsymbol{w}$ will then *lie on that same line.* If the desired vector $\boldsymbol{b}$ is *off the line*, then the equations $c\boldsymbol{v} + d\boldsymbol{w} = \boldsymbol{b}$ have no solution:

$$\textbf{Example} \qquad \boldsymbol{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \boldsymbol{w} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} \qquad \boldsymbol{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \begin{aligned} \mathbf{1}c + \mathbf{3}d &= \mathbf{1} \\ \mathbf{2}c + \mathbf{6}d &= \mathbf{0} \end{aligned}$$

Those two equations can't be solved. To eliminate the 2 in the second equation, we multiply equation 1 by 2. Then elimination subtracts $2c + 6d = \mathbf{2}$ from the second equation $2c + 6d = \mathbf{0}$. **The result is $\mathbf{0 = -2}$:** *impossible.* We not only eliminated c, we also eliminated d.

With $\boldsymbol{v}$ and $\boldsymbol{w}$ on the same line, combinations of $\boldsymbol{v}$ and $\boldsymbol{w}$ fill that line but not a plane. When $\boldsymbol{b}$ is not on that line, no combination of $\boldsymbol{v}$ and $\boldsymbol{w}$ equals $\boldsymbol{b}$. The original vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ are **"linearly dependent"** because $\boldsymbol{w} = 3\boldsymbol{v}$.

Linear combinations of two *independent vectors* $\boldsymbol{v}$ and $\boldsymbol{w}$ in two-dimensional space can produce any vector $\boldsymbol{b}$ in that plane. Then these equations have a solution:

$$\begin{matrix} \text{2 equations} \\ \text{2 unknowns} \end{matrix} \qquad c\boldsymbol{v} + d\boldsymbol{w} = \boldsymbol{b} \qquad \begin{aligned} cv_1 + dw_1 &= b_1 \\ cv_2 + dw_2 &= b_2 \end{aligned}$$

Summary The combinations $c\boldsymbol{v} + d\boldsymbol{w}$ fill the x-y plane unless $\boldsymbol{v}$ is in line with $\boldsymbol{w}$.

Important Here is a different example where elimination seems to be in trouble. But we can easily fix the problem and go on.

$$c\begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix} + d\begin{bmatrix} \mathbf{2} \\ \mathbf{3} \end{bmatrix} = \begin{bmatrix} \mathbf{2} \\ \mathbf{7} \end{bmatrix} \quad \text{or} \quad \begin{aligned} 0 + 2d &= 2 \\ c + 3d &= 7 \end{aligned}$$

That zero looks dangerous. But we only have to **exchange equations to find d and c:**

$$\begin{aligned} c + 3d &= 7 \\ 0 + 2d &= 2 \end{aligned} \quad \text{leads to} \quad d = 1 \text{ and } c = 4$$

Vectors in Three Dimensions

Suppose $\boldsymbol{v}$ and $\boldsymbol{w}$ have three components instead of two. Now they are vectors in three-dimensional space (which we will soon call $\mathbf{R}^3$). We still think of points in the space and arrows out from the zero vector. And we still have linear combinations of $\boldsymbol{v}$ and $\boldsymbol{w}$:

$$\boldsymbol{v} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} \qquad \boldsymbol{w} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad \boldsymbol{v} + \boldsymbol{w} = \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} \qquad c\boldsymbol{v} + d\boldsymbol{w} = \begin{bmatrix} 2c + d \\ 3c + d \\ c + 0 \end{bmatrix}$$

But there is a difference! The combinations of $\boldsymbol{v}$ and $\boldsymbol{w}$ **do not fill** the whole 3-dimensional space. If we only have 2 vectors, their combinations can at most fill a 2-dimensional plane. It is not a case of linear dependence, it is just a case of not enough vectors.

We need *three* independent vectors if we want their combinations to fill 3-dimensional space. Here is the simplest choice $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ for those three independent vectors:

$$\boldsymbol{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \boldsymbol{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \boldsymbol{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad c\boldsymbol{i} + d\boldsymbol{j} + e\boldsymbol{k} = \begin{bmatrix} c \\ d \\ e \end{bmatrix} \tag{1}$$

Now $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ go along the x, y, z axes in three-dimensional space $\mathbf{R}^3$. We can easily write any vector $\boldsymbol{v}$ in $\mathbf{R}^3$ as a combination of $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$:

Vector form

Matrix form

$$\boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = v_1\boldsymbol{i} + v_2\boldsymbol{j} + v_3\boldsymbol{k} \qquad \begin{bmatrix} \mathbf{1} & 0 & 0 \\ 0 & \mathbf{1} & 0 \\ 0 & 0 & \mathbf{1} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

That is the 3 by 3 **identity matrix** $\boldsymbol{I}$. Multiplying by I leaves every vector unchanged. I is the matrix analog of the number 1, because $I\boldsymbol{v} = \boldsymbol{v}$ for every $\boldsymbol{v}$.

Notice that we could not take a linear combination of a 2-dimensional vector $\boldsymbol{v}$ and a 3-dimensional vector $\boldsymbol{w}$. They are not in the same space.

How Do We Know It is a Plane?

Suppose $\boldsymbol{v}$ and $\boldsymbol{w}$ are nonzero vectors with three components each. Assume they are *independent*, so the vectors point in different directions: $\boldsymbol{w}$ is not a multiple $c\boldsymbol{v}$. *Then their linear combinations fill a plane inside 3-dimensional space.* The surface is flat. Here is one way to see that this is true:

> Look at any two combinations $c\boldsymbol{v} + d\boldsymbol{w}$ and $C\boldsymbol{v} + D\boldsymbol{w}$. Halfway between those points is $\boldsymbol{h} = \frac{1}{2}(c + C)\boldsymbol{v} + \frac{1}{2}(d + D)\boldsymbol{w}$. This is another combination of $\boldsymbol{v}$ and $\boldsymbol{w}$. So our surface has the basic property of a plane: Halfway between any two points on the surface is **another point $\boldsymbol{h}$ on the surface**. The surface must be flat!
>
> Maybe that reasoning is not complete, even with the exclamation point. We depended on intuition for the properties of a plane. Another proof is coming in Section 1.2.

Problem Set 1.1

1 Under what conditions on a, b, c, d is $\begin{bmatrix} c \\ d \end{bmatrix}$ a multiple m of $\begin{bmatrix} a \\ b \end{bmatrix}$? Start with the two equations $c = ma$ and $d = mb$. By eliminating m, find one equation connecting a, b, c, d. You can assume no zeros in these numbers.

2 Going around a triangle from $(0, 0)$ to $(5, 0)$ to $(0, 12)$ to $(0, 0)$, what are those three vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$? What is $\boldsymbol{u} + \boldsymbol{v} + \boldsymbol{w}$? What are their lengths $||\boldsymbol{u}||$ and $||\boldsymbol{v}||$ and $||\boldsymbol{w}||$? The length squared of a vector $\boldsymbol{u} = (u_1, u_2)$ is $||\boldsymbol{u}||^2 = u_1^2 + u_2^2$.

Problems 3–10 are about addition of vectors and linear combinations.

3 Describe geometrically (line, plane, or all of $\mathbf{R}^3$) all linear combinations of

$$\text{(a)} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \text{ and } \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} \qquad \text{(b)} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} \qquad \text{(c)} \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} \text{ and } \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$$

4 Draw $\boldsymbol{v} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ and $\boldsymbol{w} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$ and $\boldsymbol{v} + \boldsymbol{w}$ and $\boldsymbol{v} - \boldsymbol{w}$ in a single xy plane.

5 If $\boldsymbol{v} + \boldsymbol{w} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$ and $\boldsymbol{v} - \boldsymbol{w} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$, compute and draw the vectors $\boldsymbol{v}$ and $\boldsymbol{w}$.

6 From $\boldsymbol{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\boldsymbol{w} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, find the components of $3\boldsymbol{v} + \boldsymbol{w}$ and $c\boldsymbol{v} + d\boldsymbol{w}$.

7 Compute $\boldsymbol{u} + \boldsymbol{v} + \boldsymbol{w}$ and $2\boldsymbol{u} + 2\boldsymbol{v} + \boldsymbol{w}$. How do you know $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ lie in a plane?

These lie in a plane because $\boldsymbol{w} = c\boldsymbol{u} + d\boldsymbol{v}$. Find c and d

$$\boldsymbol{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \boldsymbol{v} = \begin{bmatrix} -3 \\ 1 \\ -2 \end{bmatrix} \quad \boldsymbol{w} = \begin{bmatrix} 2 \\ -3 \\ -1 \end{bmatrix}.$$

8 Every combination of $\boldsymbol{v} = (1, -2, 1)$ and $\boldsymbol{w} = (0, 1, -1)$ has components that add to _____. Find c and d so that $c\boldsymbol{v} + d\boldsymbol{w} = (3, 3, -6)$. Why is $(3, 3, 6)$ impossible?

9 In the xy plane mark all nine of these linear combinations:

$$c\begin{bmatrix} 2 \\ 1 \end{bmatrix} + d\begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{with} \quad c = 0, 1, 2 \quad \text{and} \quad d = 0, 1, 2.$$

10 (Not easy) How could you decide if the vectors $\boldsymbol{u} = (1, 1, 0)$ and $\boldsymbol{v} = (0, 1, 1)$ and $\boldsymbol{w} = (a, b, c)$ are linearly independent or dependent?

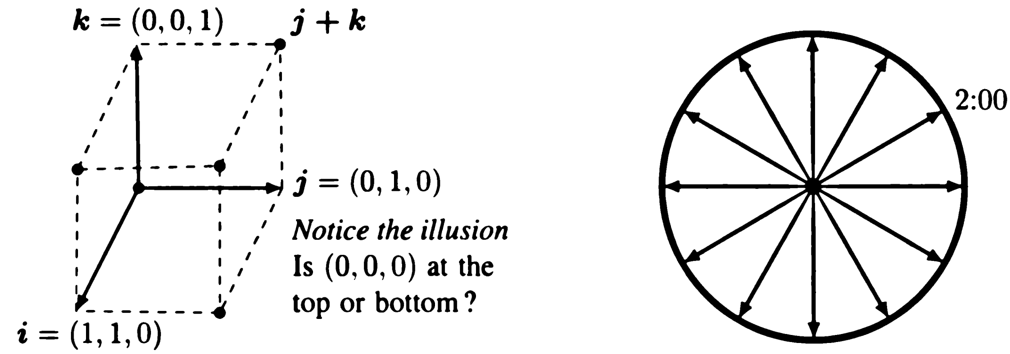


Figure 1.1: Unit cube from $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ and twelve clock vectors: all lengths = 1.

11 If three corners of a parallelogram are $(1,1)$, $(4,2)$, and $(1,3)$, what are all three of the possible fourth corners? Draw those three parallelograms.

Problems 12–15 are about special vectors on cubes and clocks in Figure 1.1.

12 Four corners of this unit cube are $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(0,0,1)$. What are the other four corners? Find the coordinates of the center point of the cube. The center points of the six faces have coordinates ____. The cube has how many edges?

13 *Review Question.* In xyz space, where is the plane of all linear combinations of $\boldsymbol{i} = (1,0,0)$ and $\boldsymbol{i} + \boldsymbol{j} = (1,1,0)$?

14 (a) What is the sum V of the twelve vectors that go from the center of a clock to the hours 1:00, 2:00, ..., 12:00?

(b) If the 2:00 vector is removed, why do the 11 remaining vectors add to 8:00?

(c) The components of that 2:00 vector are $\boldsymbol{v} = (\cos\theta, \sin\theta)$? What is θ?

15 Suppose the twelve vectors start from 6:00 at the bottom instead of $(0,0)$ at the center. The vector to 12:00 is doubled to $(0,2)$. The new twelve vectors add to ____.

16 Draw vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ so that their combinations $c\boldsymbol{u} + d\boldsymbol{v} + e\boldsymbol{w}$ fill only a line. Find vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ in 3D so that their combinations $c\boldsymbol{u} + d\boldsymbol{v} + e\boldsymbol{w}$ fill only a plane.

17 What combination $c\begin{bmatrix}1\\2\end{bmatrix} + d\begin{bmatrix}3\\1\end{bmatrix}$ produces $\begin{bmatrix}14\\8\end{bmatrix}$? Express this question as two equations for the coefficients c and d in the linear combination.

Problems 18–19 go further with linear combinations of $\boldsymbol{v}$ and $\boldsymbol{w}$ (see Figure 1.2a).

18 Figure 1.2a shows $\frac{1}{2}\boldsymbol{v} + \frac{1}{2}\boldsymbol{w}$. Mark the points $\frac{3}{4}\boldsymbol{v} + \frac{1}{4}\boldsymbol{w}$ and $\frac{1}{4}\boldsymbol{v} + \frac{1}{4}\boldsymbol{w}$ and $\boldsymbol{v} + \boldsymbol{w}$. Draw the line of all combinations $c\boldsymbol{v} + d\boldsymbol{w}$ that have $c + d = 1$.

19 Restricted by $0 \le c \le 1$ and $0 \le d \le 1$, shade in all the combinations $c\boldsymbol{v} + d\boldsymbol{w}$. Restricted only by $c \ge 0$ and $d \ge 0$ draw the "cone" of all combinations $c\boldsymbol{v} + d\boldsymbol{w}$.

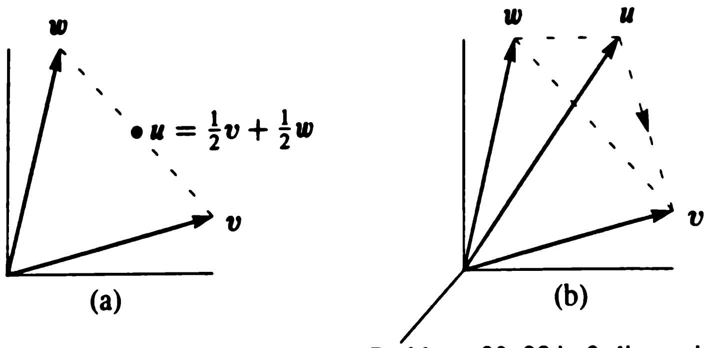


Figure 1.2: Problems **18–19** in a plane Problems **20–23** in 3-dimensional space

Problems 20–23 deal with $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ in three-dimensional space (see Figure 1.2b).

20 Locate $\frac{1}{3}\boldsymbol{u} + \frac{1}{3}\boldsymbol{v} + \frac{1}{3}\boldsymbol{w}$ and $\frac{1}{2}\boldsymbol{u} + \frac{1}{2}\boldsymbol{w}$ in Figure 1.2b. Challenge problem: Under what restrictions on c, d, e, will the combinations $c\boldsymbol{u} + d\boldsymbol{v} + e\boldsymbol{w}$ fill in the dashed triangle? To stay in the triangle, one requirement is $c \geq 0, d \geq 0, e \geq 0$.

21 The three dashed lines in the triangle are $\boldsymbol{v} - \boldsymbol{u}$ and $\boldsymbol{w} - \boldsymbol{v}$ and $\boldsymbol{u} - \boldsymbol{w}$. Their sum is _____. Draw the head-to-tail addition around a plane triangle of $(3, 1)$ plus $(-1, 1)$ plus $(-2, -2)$.

22 Shade in the pyramid of combinations $c\boldsymbol{u} + d\boldsymbol{v} + e\boldsymbol{w}$ with $c \geq 0, d \geq 0, e \geq 0$ and $c + d + e \leq 1$. Mark the vector $\frac{1}{2}(\boldsymbol{u} + \boldsymbol{v} + \boldsymbol{w})$ as inside or outside this pyramid.

23 If you look at *all* combinations of those $\boldsymbol{u}, \boldsymbol{v}$, and $\boldsymbol{w}$, is there any vector that can't be produced from $c\boldsymbol{u} + d\boldsymbol{v} + e\boldsymbol{w}$? Different answer if $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ are all in _____.

Challenge Problems

24 How many corners $(\pm 1, \pm 1, \pm 1, \pm 1)$ does a cube of side 2 have in 4 dimensions? What is its volume? How many 3D faces? How many edges? Find one edge.

25 Find *two different combinations* of the three vectors $\boldsymbol{u} = (1, 3)$ and $\boldsymbol{v} = (2, 7)$ and $\boldsymbol{w} = (1, 5)$ that produce $\boldsymbol{b} = (0, 1)$. Slightly delicate question: If I take any three vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ in the plane, will there always be two different combinations that produce $\boldsymbol{b} = (0, 1)$?

26 The linear combinations of $\boldsymbol{v} = (a, b)$ and $\boldsymbol{w} = (c, d)$ fill the plane unless _____. Find four vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}, \boldsymbol{z}$ with four nonzero components each so that their combinations $c\boldsymbol{u} + d\boldsymbol{v} + e\boldsymbol{w} + f\boldsymbol{z}$ produce all vectors in four-dimensional space.

27 Write down three equations for c, d, e so that $c\boldsymbol{u} + d\boldsymbol{v} + e\boldsymbol{w} = \boldsymbol{b}$. Write this also as a matrix equation $A\boldsymbol{x} = \boldsymbol{b}$. Can you somehow find c, d, e for this $\boldsymbol{b}$?

$$\boldsymbol{u} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \quad \boldsymbol{v} = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} \quad \boldsymbol{w} = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} \quad \boldsymbol{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

1.2 Lengths and Angles from Dot Products

1 The "dot product" of $\boldsymbol{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\boldsymbol{w} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$ is $\boldsymbol{v} \cdot \boldsymbol{w} = (1)(4) + (2)(6) = 4 + 12 = 16$.

2 The length squared of $\boldsymbol{v} = (1,3,2)$ is $\boldsymbol{v} \cdot \boldsymbol{v} = 1 + 9 + 4 = 14$. The length is $||\boldsymbol{v}|| = \sqrt{14}$.

3 $\boldsymbol{v} = (1,3,2)$ is **perpendicular** to $\boldsymbol{w} = (4,-4,4)$ because $\boldsymbol{v} \cdot \boldsymbol{w} = 0$.

4 The angle $\theta = 45°$ between $\boldsymbol{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\boldsymbol{w} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ has $\cos\theta = \dfrac{\boldsymbol{v} \cdot \boldsymbol{w}}{||\boldsymbol{v}|| \, ||\boldsymbol{w}||} = \dfrac{1}{(1)(\sqrt{2})}$.

5 All angles have $|\cos\theta| \leq 1$. All vectors have **Schwarz inequality** $|\boldsymbol{v} \cdot \boldsymbol{w}| \leq ||\boldsymbol{v}|| \, ||\boldsymbol{w}||$ | **Triangle inequality** $||\boldsymbol{v}+\boldsymbol{w}|| \leq ||\boldsymbol{v}|| + ||\boldsymbol{w}||$.

The most useful multiplication of vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ is their **dot product** $\boldsymbol{v} \cdot \boldsymbol{w}$. We multiply the first components $v_1 w_1$ and the second components $v_2 w_2$ and so on. Then we **add those results** to get a single number $\boldsymbol{v} \cdot \boldsymbol{w}$:

$$\boxed{\text{The dot product of } \boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \text{ and } \boldsymbol{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \text{ is } \boldsymbol{v} \cdot \boldsymbol{w} = v_1 w_1 + v_2 w_2.} \qquad (1)$$

If the vectors are in n-dimensional space with n components each, then

$$\textbf{Dot product} \qquad \boxed{\boldsymbol{v} \cdot \boldsymbol{w} = v_1 w_1 + v_2 w_2 + \cdots + v_n w_n = \boldsymbol{w} \cdot \boldsymbol{v}} \qquad (2)$$

The dot product $\boldsymbol{v} \cdot \boldsymbol{v}$ tells us the squared length $||\boldsymbol{v}||^2 = v_1^2 + \cdots + v_n^2$ of a vector. In two dimensions, this is the Pythagoras formula $a^2 + b^2 = c^2$ for a right triangle. The sides have $a^2 = v_1^2$ and $b^2 = v_2^2$. The hypotenuse has $||\boldsymbol{v}||^2 = v_1^2 + v_2^2 = a^2 + b^2$.

To reach n dimensions, we can add one dimension at a time. Figure 1.2 shows $\boldsymbol{v} = (1,2)$ in two dimensions and $\boldsymbol{w} = (1,2,3)$ in three dimensions. Now the right triangle has sides $(1,2,0)$ and $(0,0,3)$. Those vectors add to $\boldsymbol{w}$. The first side is in the xy plane, the second side goes up the perpendicular z axis . For this triangle in 3D with hypotenuse $\boldsymbol{w} = (1,2,3)$, the law $a^2 + b^2 = c^2$ becomes $(1^2 + 2^2) + (3^2) = 14 = ||\boldsymbol{w}||^2$.

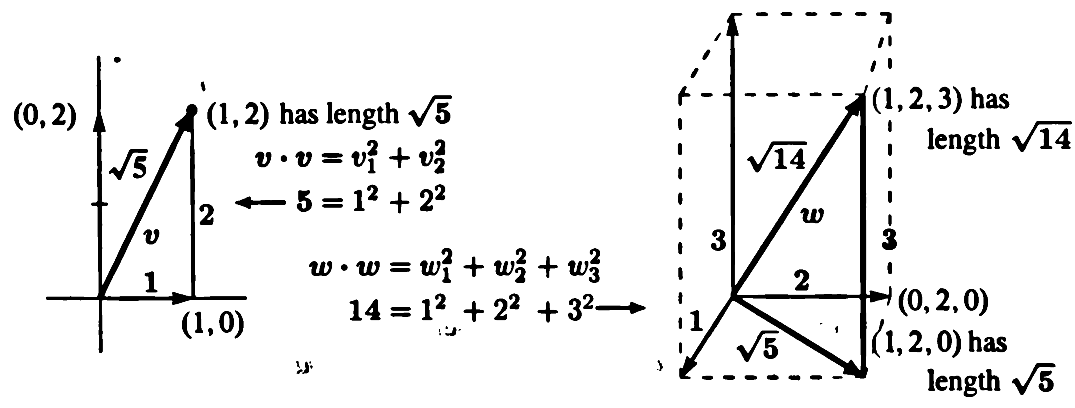


Figure 1.3: The length $\sqrt{\boldsymbol{v} \cdot \boldsymbol{v}} = \sqrt{5}$ in a plane and $\sqrt{\boldsymbol{w} \cdot \boldsymbol{w}} = \sqrt{14}$ in three dimensions.