D. Sundararajan

Fourier Analysis

A Signal Processing Approach

Second Edition

Springer

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Preface to the Second Edition

In practice, the other three versions of the Fourier analysis are approximated using the discrete Fourier transform (DFT). The amplitude profile of practical signals is usually arbitrary. Therefore, the numerical approximation of Fourier analysis is essential in practice. However, this important feature is not given due importance in the literature. This procedure has already been emphasized in the first edition for 1-D signals. In the second edition, this feature has been extended to the 2-D Fourier analysis also. Further, the approximation of Fourier analysis in the practical implementation of such important operations, such as convolution and correlation, is also emphasized. Practically biased presentation of the topics is a key feature of both the editions of this book.

The salient points of this edition include: (i) updation of some sections; (ii) additional examples; (iii) additional exercises; and (iv) some corrections.

New topics covered in this edition include: (i) sampling of bandpass signals; (ii) circular convolution from linear convolution; and (iii) more coverage of 2-D Fourier analysis.

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Preface to the First Edition

Transform methods dominate the study of linear time-invariant systems in all areas of science and engineering, such as circuit theory, signal/image processing, communications, controls, vibration analysis, remote sensing, biomedical systems, optics, acoustics, etc. The heart of the transform methods is Fourier analysis. Several other often used transforms are generalizations or specific versions of Fourier analysis. It is unique in that it is much used in theoretical studies as well as in practice. The reason for the latter case is the availability of fast algorithms to approximate the Fourier spectrum adequately. For example, the existence and continuing growth of digital signal and image processing is due to the ability to implement the Fourier analysis quickly by digital systems. This book is written for engineering, computer science and physics students, and engineers and scientists. Therefore, Fourier analysis is presented primarily using physical explanations with waveforms and/or examples, keeping the mathematical form to the extent it is necessary for its practical use. In engineering applications of Fourier analysis, its interpretation and use are relatively more important than rigorous proofs. Plenty of examples, figures, tables, programs and physical explanations make it easy for the reader to get a good grounding in the basics of Fourier signal representation and its applications.

This book is intended to be a textbook for senior undergraduate and graduate level Fourier analysis courses in engineering and science departments and a supplementary textbook for a variety of application courses in science and engineering, such as circuit theory, communications, signal processing, controls, remote sensing, image processing, medical analysis, acoustics, optics and vibration analysis. For engineering professionals, this book will be useful for self-study. In addition, this book will be a reference for anyone, student or professional, specializing in practical applications of Fourier analysis. The prerequisite for reading this book is a good knowledge of calculus, linear algebra, signals and systems, and programming at the undergraduate level.

Programming is an important component in learning and practicing Fourier analysis. A set of MATLAB® programs are available at the website of the book. While the use of a software package is inevitable in most applications, it is better to use the software in addition to self-developed programs. The effective use of a software package or to develop own programs requires a good grounding in the basic principles of the Fourier analysis. Answers to selected exercises marked $*$ are given at the end of the book. A Solutions Manual and slides are available for instructors at the website of the book.

I assume the responsibility for all the errors in this book and would very much appreciate receiving readers' suggestions and pointing out any errors (email:d_sundararajan@yahoo.com). I am grateful to my Editor and the rest of the team at Springer for their help and encouragement in completing this project. I thank my family for their support during this endeavor.

D. Sundararajan

Contents

Abbreviations

DC	Sinusoid with frequency zero, constant
DFT	Discrete Fourier transform
DIF	Decimation in frequency
DIT	Decimation in time
DTFT	Discrete-time Fourier transform
FIR	Finite impulse response
FS	Fourier series
FT	Fourier transform
IDFT	Inverse discrete Fourier transform
IFT	Inverse Fourier transform
Im	Imaginary part of a complex number or expression
LSB	Least significant bit
LTI	Linear time-invariant
PM	Plus-minus
QAM	Quadrature amplitude modulation
RDFT	DFT of real-valued data
Re	Real part of a complex number or expression
RIDFT	IDFT of the transform of real-valued data
1-D	One-dimensional
2-D	Two-dimensional

Chapter 1
Signals

Abstract Basic signals, their classifications, and shifting and scaling of signals are reviewed. First, basic signals such as impulse, sinusoid, step, ramp, and exponential are introduced. Then, the classification of signals, based on periodicity, sampling along the time and amplitude scales, even and odd symmetries, power and energy, and randomness are presented. Signal operations shifting and scaling are then explored. Finally, a review of the complex number system concludes the chapter.

Keywords Impulse · Sinusoid · Exponential · Unit-step · Even symmetry · Odd symmetry · Energy · Power · Time-shifting · Time reversal · Signal expansion · Signal compression · Complex numbers

Signals convey some information. Signals are abundant in the applications of science and engineering. Typical signals are audio, video, biomedical, seismic, radar, vibration, communication, and sonar. While the signals are mostly of continuous nature, they are usually converted to digital form and processed by digital systems for efficiency. In signal processing, signals are enhanced to improve their quality with some respect, or some features are extracted or they are modified in some desired way. A signal, in its mathematical representation, is a function of one or more independent variables. While time is the independent variable most often, it could be anything else, such as distance. The analysis is equally applicable to all types of independent variables. Signal or waveform is used to refer to the physical form of a signal. In its mathematical representation, a signal is referred to as a function or sequence. This usage is not strictly adhered.

The amplitude profile of most naturally occurring signals is arbitrary and, consequently, it is difficult to analyze, interpret, transmit, and store them in their original form. The idea of a transform is to represent a signal in an alternative, but equivalent, form to gain advantages in its processing. Fourier analysis, the topic of this book, provides a widely used representation of signals. As signal

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representation is important and the most suitable representation of a signal depends on its characteristics, we have to first study the classification of signals. Further, the representation is in terms of some well-defined basis signals, such as the sinusoid, complex exponential, and impulse. Although practical signals are mostly real-valued, it becomes mandatory to use the equivalent complex-valued signals for ease of mathematical manipulation. In addition, operations such as shifting and scaling of signals are often required in signal analysis. All these aspects are presented in this chapter.

1.1 Basic Signals

The amplitude profile of most naturally occurring signals is arbitrary. These signals are analyzed using some well-defined basic signals, such as the impulse, step, ramp, sinusoidal, and exponential signals. In addition, systems, which are hardware or software realizations, modify signals or extract information from them. They are also characterized by their responses to these signals. The basic signals either have an infinite duration or infinite bandwidth. For practical purposes, they are approximated to a desired accuracy. Fourier analysis has four versions and each version represents different type of signals in the frequency domain. Therefore, it is necessary to study both the continuous and discrete type of signals.

1.1.1 Unit-Impulse Signal

The unit-impulse and the sinusoidal signals are the most important signals in the study of signals and systems. The continuous unit-impulse $\delta(t)$ is a signal with a shape and amplitude such that its integral at the point $t = 0$ is unity. It is defined, in terms of an integral, as

$$\int_{-\infty}^{\infty} x(t)\delta(t)\,dt = x(0)$$

It is assumed that $x(t)$ is continuous at $t = 0$ so that the value $x(0)$ is distinct. The product of $x(t)$ and $\delta(t)$ is

$$x(t)\delta(t) = x(0)\delta(t)$$

since the impulse exists only at $t = 0$. Therefore,

$$\int_{-\infty}^{\infty} x(t)\delta(t)\,dt = x(0)\int_{-\infty}^{\infty} \delta(t)\,dt = x(0)$$

The value of the function $x(t)$, at $t = 0$, is sifted out or sampled by the defining operation. By using shifted impulses, any value of $x(t)$ can be sifted.

It is obvious that the integral of the unit-impulse is the unit-step $u(t)$. Therefore, the derivative of the unit-step signal is the unit-impulse signal. The value of the unit-step is zero for $t < 0$ and 1 for $t > 0$. Therefore, the unit area of the unit-impulse, as the derivative of the unit-step, must occur at $t = 0$. The unit-impulse and the unit-step signals enable us to represent and analyze signals with discontinuities as we do with continuous signals. For example, these signals model the commonly occurring situations such as opening and closing of switches.

The continuous unit-impulse $\delta(t)$ is difficult to visualize and impossible to realize in practice. However, the approximation of it by some functions is effective in practice and can be used to visualize its effect on signals and its properties. While there are other functions that approach an impulse in the limit, the rectangular function is often used to approximate the impulse. The unit-impulse, for all practical purposes, is essentially a narrow rectangular pulse with unit area. Suppose we compress it by a factor of 2, the area, called its strength, becomes $1/2 = 0.5$. The scaling property of the impulse is given as

$$\delta(at) = \frac{1}{|a|}\delta(t), \ a \neq 0$$

With $a = -1$, $\delta(-t) = \delta(t)$ implies that the impulse is an even-symmetric signal. For example,

$$\delta(3t-1) = \delta\left(3\left(t - \frac{1}{3}\right)\right) = \frac{1}{3}\delta\left(t - \frac{1}{3}\right)$$

The discrete unit-impulse signal, shown in Fig. 1.1a, is defined as

$$\delta(n) = \begin{cases} 1 \text{ for } n = 0 \\ 0 \text{ for } n \neq 0 \end{cases}$$

The independent variable is n and the dependent variable is $\delta(n)$. The only nonzero value (unity) of the impulse occurs when its argument $n = 0$. The shifted impulse $\delta(n-k)$ has its only nonzero value at $n = k$. Therefore, $\sum_{n=-\infty}^{\infty} x(n)\delta(n-$

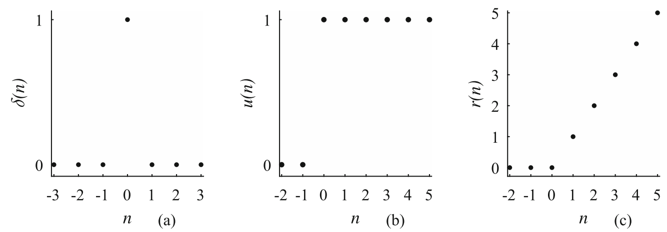

Fig. 1.1 (**a**) The discrete unit-impulse signal, $\delta(n)$; (**b**) the discrete unit-step signal, $u(n)$; and (**c**) the discrete unit-ramp signal, $r(n)$

$k) = x(k)$ is called the sampling or sifting property of the impulse. For example,

$$\sum_{n=-\infty}^{\infty} 3^n \delta(n) = 1, \ \sum_{n=0}^{2} 9^n \delta(n+1) = 0, \ \sum_{n=-2}^{0} 4^n \delta(-n-1) = 0.25,$$

$$\sum_{n=0}^{2} 2^n \delta(n-1) = 2, \ \sum_{n=-\infty}^{\infty} 2^n \delta(n+1) = 0.5, \ \sum_{n=-\infty}^{\infty} 3^n \delta(n-3) = 27$$

The argument $n + 1$ of the impulse, in the second summation, never becomes zero within the limits of the summation.

1.1.2 Unit-Step Signal

The discrete unit-step signal, shown in Fig. 1.1b, is defined as

$$u(n) = \begin{cases} 1 \text{ for } n \geq 0 \\ 0 \text{ for } n < 0 \end{cases}$$

For positive values of its argument, the value of the unit-step signal is unity and it is zero otherwise. An arbitrary function can be expressed in terms of appropriately scaled and shifted unit-step or impulse signals. By this way, any signal can be specified, for easier mathematical analysis, by a single expression, valid for all n. For example, a pulse signal, shown in Fig. 1.2a, with its only nonzero values defined as $\{x(1) = 1, x(2) = 1, x(3) = 1\}$, can be expressed as the sum of the two delayed unit-step signals shown in Fig. 1.2b, $x(n) = u(n-1) - u(n-4)$. The pulse can also be represented as a sum of delayed impulses.

$$x(n) = u(n-1) - u(n-4) = \sum_{k=1}^{3} \delta(n-k) = \delta(n-1) + \delta(n-2) + \delta(n-3)$$

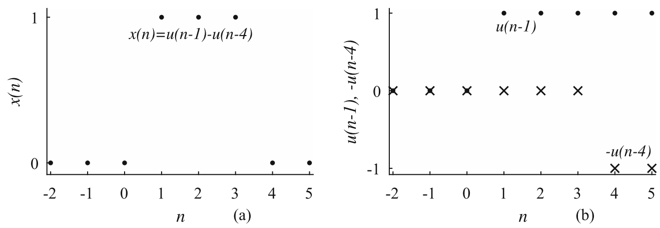

Fig. 1.2 (**a**) The pulse signal, $x(n) = u(n-1) - u(n-4)$ and (**b**) the delayed unit-step signals, $u(n-1)$ and $-u(n-4)$

The continuous unit-step signal is defined as

$$u(t) = \begin{cases} 1 & \text{for } t > 0 \\ 0 & \text{for } t < 0 \\ \text{undefined} & \text{for } t = 0 \end{cases}$$

The value $u(0)$ is undefined and can be assigned a suitable value from 0 to 1 to suit a specific problem. In Fourier analysis, $u(0) = 0.5$. A common application of the unit-step signal is that multiplying a signal with it yields the causal form of the signal. For example, the continuous signal $\sin(t)$ is defined for $-\infty < t < \infty$. The values of $\sin(t)u(t)$ is zero for $t < 0$ and $\sin(t)$ for $t > 0$.

1.1.3 Unit-Ramp Signal

The discrete unit-ramp signal, shown in Fig. 1.1c, is also often used in the analysis of signals and systems. It is defined as

$$r(n) = \begin{cases} n \text{ for } n \geq 0 \\ 0 \text{ for } n < 0 \end{cases}$$

It linearly increases for positive values of its argument and is zero otherwise.

The three signals, the unit-impulse, the unit-step, and the unit-ramp, are related by the operations of sum and difference. The unit-impulse signal $\delta(n)$ is equal to $u(n) - u(n-1)$, the first difference of the unit-step. The unit-step signal $u(n)$ is equal to $\sum_{k=0}^{\infty} \delta(n-k)$, the running sum of the unit-impulse. The shifted unit-step signal $u(n-1)$ is equal to $r(n) - r(n-1)$. The unit-ramp signal $r(n)$ is equal to

$$r(n) = nu(n) = \sum_{k=0}^{\infty} k\delta(n-k) = \sum_{k=-\infty}^{n-1} u(k)$$

Similar relations hold for continuous type of signals.

1.1.4 Sinusoids and Complex Exponentials

1.1.4.1 Sinusoids

The impulse and the sinusoid are the two most important signals in signal and system analysis. The impulse is the basis for convolution and the sinusoid is the basis for transfer function. The cosine and sine functions are two of the most